Math 142 Lecture 12 Notes

Daniel Raban

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1 Induced Maps and the Fundamental Group of S^1

1.1 Induced maps

Recall that given a space X and $p \in X$, we can define the fundamental group based at p

$$\pi_1(X,p) = \{[\gamma] : (\gamma : [0,1] \to X) \text{ is continuous}, \gamma(0) = \gamma(1) = p\},$$

where $[\gamma] = [\gamma']$ iff $\gamma \simeq \gamma'$ rel $\{0,1\}$.

We also showed that the basepoint did not matter if the space was path-connected (i.e. $\pi_1(X,p) \cong \pi_1(X,q)$). The idea of this proof was that we take a path γ from p to q, and given $[\alpha] \in \pi_1(X,p)$, send $[\alpha] \mapsto [\gamma^{-1} \cdot \alpha \cdot \gamma]$. This converts paths based at p to paths based at q by running γ (and its inverse) at the beginning and end of the path. We called this isomorphism γ_*; in general this map depends on γ.

Definition 1.1. If we have a continuous map $f : X \to Y$ such that $f(p) = q$, we get a homomorphism $f_* : \pi_1(X,p) \to \pi_1(Y,q)$ sending $[\alpha] \mapsto [f \circ \alpha]$. We say f_* is *induced* by f.

The proof that f_* is a homomorphism is the same as the proof that γ_* is a homomorphism, so we will not repeat it.

Theorem 1.1. *If $f : X \to Y$ and $g : Y \to Z$, then*

$$(g \circ f)_* = g_* \circ f_*.$$

Proof. This follows from the definition and properties of compositions and homotopy. □

Remark 1.1. The identitfy function $\mathrm{id}_X : X \to X$ induces an isomorphism $\pi_1(X,p) \to \pi_1(X,p)$, the identity isomorphism. So if $f : X \to Y$ is a homeomorphism, then $f_*^{-1} \circ f_* = (\mathrm{id}_X)_*$ and $f_* \circ f_*^{-1} = (\mathrm{id}_X)_*$, and we get that f_* is an isomorphism from $\pi_1(X,p) \to \pi_1(Y, f(p))$.

1.2 The fundamental groups of contractible spaces and S^1

Let's find the fundamental group of some spaces.

Example 1.1. Let X be convex (or contractible). Then $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = p$ is homotopic to $\gamma_p : [0,1] \to X$ which sends $x \mapsto p$ via the straight line homotopy.[1] Recall that this is

$$F(x,t) = (1-t)\gamma(x) + t\gamma_p(x).$$

Note that $F(0,t) = (1-t)\gamma(0) + t\gamma_p(0) = p$ and $F(1,t) = p$, so $\gamma \simeq_F \gamma_P$ rel $\{0,1\}$. So $\pi_1(X,p) \cong 1$, the trivial group.

Let $S^1 = \{e^{i\theta} \in \mathbb{C}; \theta \in \mathbb{R}\}$ be the circle. Let $f : \mathbb{R} \to S'$ be $x \mapsto e^{2\pi i x}$, and let $\gamma_n : [0,1] \to \mathbb{R}$ be $x \mapsto nx$. Then $f \circ \gamma_n [0,1] \to S^1$ mapping $x \mapsto e^{2\pi i n x}$ is a path in S^1 from 1 to 1, and it wraps around the circle $|n|$ times (counterclockwise if $n > 0$ and clockwise if $n < 0$).

Theorem 1.2. *The map $\phi : \mathbb{Z} \to \pi_1(S^1, 1)$ sending $n \mapsto [f \circ \gamma_n]$ is an isomorphism.*

Proof. First note that if $\gamma_n' : [0,1] \to \mathbb{R}$ has $\gamma_n'(0) = 0$ and $\gamma_n'(1) = n$, then $\gamma_n \simeq \gamma_n'$ rel $\{0,1\}$ (as $\mathbb{R}$ is convex). This implies that $[f \circ \gamma_n] = [f \circ \gamma_n']$.

ϕ is a homomorphism: If $m, n \in \mathbb{Z}$ let $\sigma : [0,1] \to \mathbb{R}$ send $x \mapsto \gamma_n(x) + m$. Note that $f \circ \sigma = f \circ \gamma_n$, and $\gamma_m \cdot \sigma$ is a path from 0 to $m+n$. So

$$\begin{aligned}
\phi(m+n) &= [f \circ \gamma_{m+n}] = [f \circ (\gamma_m \cdot \sigma)] \\
&= [(f \circ \gamma_m) \cdot (f \circ \sigma)] \\
&= [(f \circ \gamma_m) \cdot (f \circ \gamma_n)] \\
&= [f \circ \gamma_m][f \circ \gamma_n]
\end{aligned}$$

ϕ is surjective: We use a "path lifting" lemma: If σ is a path in S^1 beginning at 1, then there is a unique path $\tilde{\sigma}$ in $\mathbb{R}$ starting at 0 such that $f \circ \tilde{\sigma} = \sigma$; the map $\tilde{\sigma}$ is called a *lift* of σ. So if $\alpha \in \pi_1(S^1, 1)$, then there exists a path σ such that $\alpha = [\sigma]$. From the lemma, there exists a unique path $\tilde{\sigma} : [0,1] \to \mathbb{R}$ with $\tilde{\sigma}(0) = 0$ and $f \circ \tilde{\sigma} = \sigma$. So $[f \circ \tilde{\sigma}] = \alpha$, and then $\phi(\tilde{\sigma}(1)) = \alpha$. So ϕ is surjective.

ϕ is injective: We use a "homotopy lifting" lemma: If σ, σ' are paths from 1 to 1 in S^1 with $\sigma \simeq_F \sigma'$ rel $\{0,1\}$, then there exists a unique homotopy $\tilde{F}$ from $\tilde{\sigma}$ to $\tilde{\sigma}'$ such that $f \circ \tilde{F} = F$; here, $\tilde{F}$ is a lift of F. So if $\phi(n) = e \in \pi_1(S^1, 1)$, then $f \circ \gamma_n \simeq_F e$ rel $\{0,1\}$, where $e : [0,1] \to S^1$ sends $x \mapsto 1$. The lemma implies that there exists a unique homotopy $\tilde{F}$ such that $f \circ \tilde{F} = F$.

The domain of $\tilde{F}$ is the square $[0,1] \times [0,1]$. We also know that $F(0,t) = F(1,t) = 1$ for all $t \in [0,1]$ and that $F(x,1) = e(x) = 1$ for all $x \in [0,1]$. So if P is the union of the left, top, and right edges of the square, then $F(P) = 1$. Then $\tilde{F}(P) \subseteq \mathbb{Z}$. But P is connected,

[1] In a non-convex but contractible space, you may have to use a different homotopy.

and $\mathbb{Z}$ is discrete, so $\tilde{F}(P)$ is a singleton. Observe that $\sigma(0) = \tilde{\sigma}(0) = 0$, so $\tilde{F}(0,0) = 0$; then $\tilde{F}(P) = \{0\}$. Also, $F(x,0) = \gamma_n$, as the lift is unique. So $n = \gamma_n(1) = \tilde{F}(1,0) = 0$. So $\phi(n) = e$ implies that $n = 0$, making ϕ injective. $\square$